An Analytic Orbit Prediction Program Generator

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An ephemeris program generator is designed to automatically construct compact efficient ephemeris programs tailored specifically for tasks encountered in online real-time computing environments. For portability the ephemeris programs are written in standard FORTRAN IV. The analytic theory underlying the ephemeris program is the theory developed by Alfriend and Coffey which solves the zonal satellite problem is closed form. The flexibility required by a diverse user community is built into a meta-program which automatically constructs the ephemeris programs according to a program template. The program template is dictated from a terminal according to the specifications of the particular application. Using the program template the meta-program transcribes the series representing the theory into FORTRAN-compilable code. The largest version of the current ephemeris program can be held in the memory of current microcomputers and could be adapted for use in multiprocessing or distributed processing environments.

I. Introduction

THE analytic orbit prediction program generator is designed to construct a set of compact efficient ephemeris programs automatically, each one tailored for a specific task such as might be encountered in online real-time applications.

The concept of the meta-program is illustrated best in Fig. 1. The meta-program, resident on a large general-purpose computer, serves to filter the theory files into the ephemeris programs. The generated programs may be targeted for a variety of computing environments. Guided by a user-supplied program template, the meta-program accesses the theory data base, consisting of first- and second-order coordinate transformations and secular terms through the third order, to construct the ephemeris program. The basic analytic theory underlying the ephemeris program is the theory of Coffey and Deprit[1] as modified by Alfriend and Coffey.[2] This theory solves the zonal satellite problem in closed form. The theory implemented herein includes the zonal harmonics J_2-J_4.

Two objectives were in mind when the ephemeris program was planned. The authors wanted to provide a program that would be useful to a large number of users, a sort of general-purpose program, within the scope of the analytic theory, that would provide a range of options allowing the user to select relevant parts of the theory to fit a particular problem.

It was also desired that the program be compact enough and fast enough that it could run on current desk top microcomputers. This would provide researchers with a very accurate analytic satellite theory that could be used as a tool much like that currently used in preprogrammed packages on hand-held calculators. Such a program also could be adapted for online real-time orbit computations such as are performed in the computers used to drive antennas and used on board satellites.

These objectives tend to be mutually exclusive because of the large overhead required to support a general-purpose program and because usually only a small percentage of a general-purpose program is used for a given application. To satisfy both of these objectives the authors designed a parent program, dubbed a meta-program, which constructs an ephemeris program automatically, according to a program template dictated during a short interactive session at a terminal. The meta-program provides all of the options that usually would be included in a general-purpose program. Following the instructions of the program template, the meta-program transcribes the series representing the requested parts of the theory into FORTRAN-compilable code and generates all of the necessary subroutines for the ephemeris program. In particular, the extent of arrays and do-loops, which vary from application to application, are computed by the meta-program. Evaluation of those parts of the series, dependent only on physical constants, is performed by the meta-program and transferred as data initialization statements to the ephemeris program. In short, the product of the meta-program is an ephemeris program tailored to a particular application that is complete and ready to compile.

To facilitate the programming of the meta-program, the authors chose to code it in a pseudo-FORTRAN language called RATFOR.[3] RATFOR, which stands for Rational FORTRAN, represents both a high-level programming language that encourages structured programming techniques, and a program preprocessor that translates RATFOR code into standard FORTRAN IV statements.

The meta-program provides a number of benefits. The theory itself is complex, involving 51 distinct Poisson series representing the direct and inverse equations for three canonical transformations. It was anticipated that some errors might exist in the theory that would be uncovered only by comparing ephemerides computed from the theory with ephemerides computed by other independent orbit programs. In fact, one such error was discovered in the theory. In addition the meta-program enabled us to revise the construction of the theory easily to accommodate a number of simplifications resulting in improved computation time. Since the purpose of the meta-program is to filter the theory into an ephemeris program, a change in the theory is automatically reflected in the next generation of the ephemeris programs.

To provide a reasonable amount of completeness and to describe the differences adequately between the ephemeris programs, a brief description of the development of the theory is provided in Sec. II. Section III is devoted to programming techniques used to minimize both memory requirements and execution times of the ephemeris programs. In Sec. IV several procedures are described that were used to establish the accuracy of the ephemeris programs and to test the correctness of the implementations of the analytical

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theory. To reduce the long-term secular drift in position, the averaged mean motions are computed to one order higher than the coordinate transformations for each implementation of the theory. This enabled the ephemeris programs to maintain a high degree of accuracy over several hundred orbits.

All versions of the ephemeris programs described in this paper are available by contacting the first author at the Naval Research Laboratory.

II. Formulation of the Theory

The Hamiltonian for a satellite perturbed by the zonal harmonics from the potential of the Earth is given by

$$H=H(r,\theta,\nu,R,\Theta,N)=H_0+\epsilon H_1+\epsilon^2\sum_{n\geq 2}H_n \qquad (1)$$

where

$$H_0=\frac{1}{2}\left(R^2+\frac{\Theta^2}{r^2}\right)-\frac{\mu}{r}$$

$$H_1=-\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2 P_2(s\sin\theta)$$

$$H_n=-\frac{J_n}{J_2^2}\frac{\mu}{r}\left(\frac{\alpha}{r}\right)^n P_n(s\sin\theta)$$

$$s=\sin i=\sqrt{1-N^2/\Theta^2}$$

This defines a Hamiltonian with two degrees of freedom. The right ascension of the node ν is ignorable, and $\theta=f+g$ is the argument of latitude. The variable r is the radial distance from the Earth's center of mass to the satellite. The variables R, Θ, and N are the momenta for r, θ, and ν, respectively. The constants μ and α stand for the Keplerian constant and the equatorial radius of the Earth, respectively, while i is the inclination of the satellite's orbital plane. The small parameter is $\epsilon=-J_2$. The Legendre polynomials are represented by P_n and the zonal coefficients are represented by J_n.

The theory developed by Alfriend and Coffey[2] is an extensive modification of the theory developed by Coffey and Deprit.[1] The theory involves the development of three canonical transformations of the Lie type[4] to eliminate systematically the short- and long-period terms. For implementation herein, each of these transformations has been constructed to accommodate the zonals J_2, J_3, J_4 and is developed to order 2. Of course further extensions to include other zonals can be done easily.

The first transformation in the theory, called the elimination of the parallax,[5] has the effect of converting the factors

$$\frac{\mu}{r}\left(\frac{p}{r}\right)^n \quad \text{to} \quad \frac{\mu}{p}\left(\frac{p}{r}\right)^2, \qquad \Theta^2=\frac{\mu}{p}$$

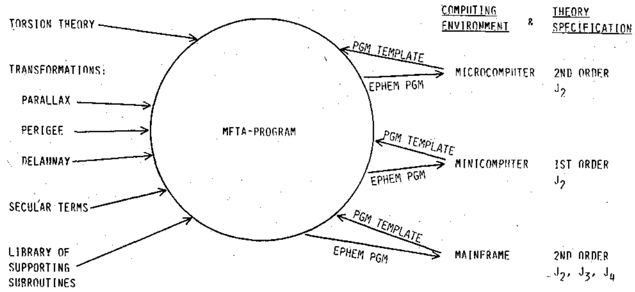

Fig. 1 Meta program concept.

while eliminating the true anomaly from the angle θ. The resulting Hamiltonian takes the form

$$H=H_0+\sum_{n\geq 1}\frac{\epsilon^n}{n!}H_n$$

where

$$H_0=\frac{1}{2}\left(R^2+\frac{\Theta^2}{r^2}\right)-\frac{\mu}{r}$$

$$H_1=\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^2\left(\frac{1}{2}-\frac{3}{4}s^2\right)$$

$$H_n=\frac{\Theta^2}{r^2}\sum_{k\geq 3}F_{n,k}\left(\frac{J_k}{J_2^2},\bar{C},\bar{S},s,p\right) \qquad n\geq 2 \qquad (2)$$

The $F_{n,k}$'s are functions of the variables $\bar{C},\bar{S},s,p$ and the constants J_k/J_2^2, where k depends on the zonals included in the theory. The variables $\bar{C}$ and $\bar{S}$ are functions of the elements r,θ,R,Θ through the formulas

$$\frac{p}{r}=1+\bar{C}\cos\theta+\bar{S}\sin\theta, \quad R=\frac{\bar{C}\Theta}{p}\sin\theta-\frac{\bar{S}\Theta}{p}\cos\theta$$

In terms of the common orbital elements, $\bar{C}=e\cos g$, $\bar{S}=e\sin g$.

Throughout this paper four different Hamiltonians and different orders of each one will be discussed. To keep the notation concise, the same symbol H will be used to represent each of the Hamiltonians, the context in which it is used will identify the particualr Hamiltonian it represents.

The second transformation, called the elimination of the perigee,[2] renders the argument of the perigee an ignorable coordinate. The resulting Hamiltonian assumes the form

$$H=H_0+\sum_{n\geq 1}\frac{\epsilon^n}{n!}H_n$$

where

$$H_0=\frac{1}{2}\left(R^2+\frac{\Theta^2}{r^2}\right)-\frac{\mu}{r}$$

$$H_1=\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)\left(\frac{1}{2}-\frac{3}{4}s^2\right)$$

$$H_n=\frac{\Theta^2}{r^2}\sum_{k\geq 3}F_{n,k}\left(\frac{J_k}{J_2^2},e^2,s,p\right) \qquad n\geq 2 \qquad (3)$$

The variables $\bar{C}$ and $\bar{S}$ enter this Hamiltonian only through the combination $\bar{C}^2+\bar{S}^2=e^2$. Thus the argument of perigee is not present in this Hamiltonian.

For the third transformation, the Hamiltonian is considered as a function of the Delaunay variables (ℓ,g,h,L,G,N). The short-period terms reside in the Hamiltonian in the factor $(\Theta/r)^2=(\Theta/p)^2\ (1+e\cos f)^2$. The variable f is the true anomaly which is related to the mean anomaly ℓ through Kepler's equation. A Delaunay normalization is then constructed to reduce the Hamiltonian to the secular Hamiltonian

$$H=H_0+\sum_{n\geq 1}\frac{\epsilon^n}{n!}H_n$$

$$H_1=\frac{G^2}{p^2}\left(\frac{\alpha}{p}\right)^2\eta^3\left(\frac{1}{2}-\frac{3}{4}s^2\right)$$

$$H_n=\frac{G^2}{p^2}\sum_{k\geq 3}F_{n,k}\left(\frac{J_k}{J_2^2},e^2,s,p\right) \qquad n\geq 2$$

$$\eta=\sqrt{1-e^2} \qquad (4)$$

It is from this Hamiltonian that the secular rate of change of the angular variables ℓ, g, h are derived. Thus,

$$\frac{d\ell}{dt}=\frac{\partial H}{\partial L}, \quad \frac{dg}{dt}=\frac{\partial H}{\partial G}, \quad \frac{dh}{dt}=\frac{\partial H}{\partial N}$$

In Appendix A the authors provide the partials of this Hamiltonian through order 2.

Of course each of the new Hamiltonians, Eqs. (2-4), is developed simultaneously with a corresponding generating function

$$W=\sum_{n\geq 0}\frac{\epsilon^n}{n!}W_{n+1}$$

The generating functions retain the periodic information eliminated from the Hamiltonians, to reclaim it one must transform each of the coordinates by the generating function according to the rules specified by the theory of Lie transformations.[4] This amounts to completing the Lie triangle for each of the coordinates.

To illustrate how to obtain these transformation equations suppose we designate Δ as any of the coordinates r,θ,ν,R,Θ in the case of the parallax or the perigee transformation or as ℓ,g,h,L in the case of the Delaunay transformations. Then the transformation equation is given to order 2 by

$$\Delta=\Delta'+\epsilon(\Delta';W_1)+(\epsilon^2/2)\{(\Delta';W_1)+((\Delta';W_1);W_1)\} \tag{5}$$

To obtain the inverse of Eq. (5) one can use the inverse generating function, which to order 1 is given by

$$V=-W_1-\epsilon W_2$$

Thus the inverse equation for Eq. (5) is given by

$$\Delta'=\Delta-\epsilon(\Delta;W_1)+(\epsilon^2/2)\{-(\Delta;W_1)+((\Delta;W_1);W_1)\} \tag{6}$$

It is important to note the mixed sign in the second-order term of Eq. (6). Although the inverse generator is merely the negative of the generator, the inverse equations cannot be obtained by merely changing the signs in Eq. (5).

Since the longitude of the node ν is an ignorable coordinate, the transformation of N, the polar component of the angular momentum, is the identity for each of the three transformations. Also for the Delaunay transformation the angular momentum is an integral.

The transformation equations for the parallax, perigee, and Delaunay transformations are given to order 1 in Appendix B.

Although the theory described so far is not applicable to satellites within the Earth's drag region, it is more comprehensive than other analytic theories for a large number of satellites not affected by the atmosphere.

The meta-program also provides an option to select a first-order theory that differs from the theory described thus far. After the elimination of the parallax one can complete a first-order theory for the main problem by applying the torsion transformation developed by Deprit.[5] Due to the brevity of the torsion transformation this first-order theory can be computed in one-third the time of the previously described theory, although for most applications it is not as accurate. An interesting aspect of the torsion theory is that it does not exhibit the eccentricity and inclination singularities which exist in the theory of Alfriend and Coffey.

III. Programming Considerations

To achieve maximum portability, the ephemeris programs were designed to comply with the standard FORTRAN IV conventions. The VAX 11-780 was used as the development computer and the ephemeris programs were run on the VAX and another Laboratory computer, the Texas Instruments Advanced Scientific Computer. One of the options in the meta-program specifies the target computer for the ephemeris programs. At the request of users, the authors intend to install those modifications required by different computing environments in the meta-program.

The technique for storing and evaluating the Poisson series representing the transformations in the theory occupies a central role in the meta-program. Each term in a series is composed of the product of three essential components

$$\text{Term}=\left[\frac{\text{num}\ J_n}{\text{den}\ J_2^2}\right]\left[\prod_{i\geq 1}x_i^{\beta_i}\right]\left[\begin{matrix}\cos\\ \sin\end{matrix}(\alpha\theta)\right]$$

The first component consisting of a rational number and a physical constant is evaluated to a real number. This number then occupies one element of an array consisting of all such numbers for a particular Poisson series. A data statement which initializes the array at compile time is delivered by the meta-program to the ephemeris program. As a tradeoff between accuracy and memory goals, the meta-program provides double precision initialization for all first-order transformations and second-order secular terms and provides single-precision initialization for second-order transformations and third-order secular terms.

The information for the monomial

$$\prod_{i\geq 1}x_i^{\beta_i}$$

and the trigonometric component is recorded in separate integer*2 arrays which are again initialized via FORTRAN data statements for the ephemeris program.

At the time the data statements are being written the range of the exponents in the monomial are tracked by the meta-program. This information is used to set up, in the ephemeris program, a matrix consisting of the powers of the variables in the monomial. Similarly the range of the arguments of the angular variables is recorded by the meta-program. This information serves to set the maximum extent for sine and cosine arrays which are computed once per time step for each transformation. A fast Fourier series evaluation algorithm[6] is implemented to evaluate the trigonometric terms. Each term in a series in then calculated in a straightforward manner by multiplying together the rational coefficient, the product of the variables in the monomial as stored in the powers matrix, and the trigonometric terms.

Special consideration is paid to the evaluation of the third transformation, the Delaunay transformation. The equation

Table 1 Run time[a] statistics

Theory	Memory requirements, bytes	CPU time, s/1000 time steps
Torsion	6,144	9.4
1st-order J_2, 2nd-order secular	11,776	29.8
1st-order J_2, J_3, J_4, 2nd-order secular	12,800	39.5
2nd-order J_2, 3rd-order secular	28,672	140
2nd-order J_2, J_3, J_4, 3rd-order secular	41,984	299

[a] All of the runs indicated in Table 1 were performed on a VAX 11-780 which is reputedly capable of executing instructions at a speed of 831 KOPS (thousands of operations per second).

of the center, $\phi = f - \ell$, is the only time-varying term in the monomial component. The remainder of the monomial depends only on the constant momenta, and thus is evaluated outside of the time-varying loop.

Table 1 records some statistics on running times for several versions of the ephemeris programs. The motivation behind having separate ephemeris programs to meet different accuracy requirements is evident from this table. An increase in complexity of the theory necessarily must result in an increase of computer resources to calculate a satellite's position.

IV. Theory Validation and Numerical Results

A number of steps have been taken to ensure the correctness of the theory and the ephemeris programs. Each transformation—parallax, perigee, and Delaunay—should maintain an accuracy consistent with the order of the transformations. For example, applying one of the second-order transformations to a set of coordinates and feeding the transformed coordinates into the inverse equations should yield an error no greater than $\mathcal{O}(J_2^3)$. After correction of one theoretical error the authors were able to establish this accuracy check for the parallax and perigee transformations. One should not expect that the Delaunay transformation could pass these requirements equally well. For instance normalized errors in the mean anomaly, ℓ, and argument of perigee, g, were of the order 10^{-4} at first order and 10^{-7} at second order, although the errors for $\ell + g$ were of the order 10^{-6} and 10^{-9}, respectively. The reason is that the eccentricity is a divisor in the equations for ℓ and g but not a divisor for $\ell + g$. A small error still persists when computing the Whittaker variables, $r, \theta, \nu, R, \Theta$, from the Delaunay variables since the true anomaly must be computed by the nonlinear Kepler equation. In another check the authors established that the value of the Hamiltonian remains constant along an orbit to within the predicted accuracy of the theory.

Once the tests mentioned above were passed, the ephemerides computed by the programs were compared against the ephemerides computed by numerical integration from the Goddard Trajectory Determination System (GTDS). In the literature, much attention has been paid as to how one initializes the averaged elements for an analytic theory. It is a common practice for first-order theories to employ a differential correction procedure on the averaged coordinates to force the osculating coordinates to fit best a reference orbit over some time span. The adjusted averaged coordinates are then used to propagate the orbit. Since the primary purpose of the comparisons produced here is to validate the analytic theory, we calculate the averaged elements through the transformation equations starting from the osculating coordinates at epoch. The same set of osculating coordinates is used to propagate the orbit by the numerical integration. The following figures represent the results of several comparisons. Along the y axis in each of the figures is plotted the Euclidean normed difference in ephemerides (the square root of the sum of the squares of the difference in position in Cartesian coordinates). All of the numerical integration runs used the twelfth-order Cowell integrator with a fixed step size of 100 s.

Two different orbits were selected for comparison. All of the graphs exhibit a periodic variation centered around the expected secular growth in the difference of the ephemerides. Figures 2 and 3 compare ephemerides for a typical elliptic orbit with eccentricity $e = 0.01$, semimajor axis $a = 7000$ km, and inclination $i = 70$ deg. The period for this orbit is 1.61 h. Figure 3 illustrates the increase in accuracy provided by the second-order theory for this orbit. After about 150 orbits the difference in ephemerides provided by the second-order theory is still less than 3.5 m.

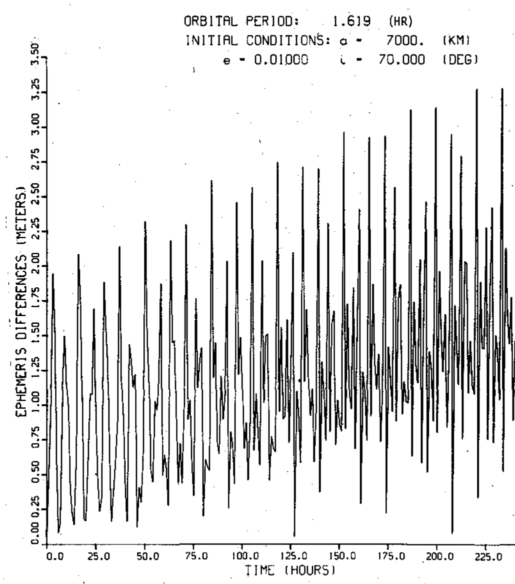

Fig. 3 GTDS (Cowell)/AOPP (order 2, J_2).

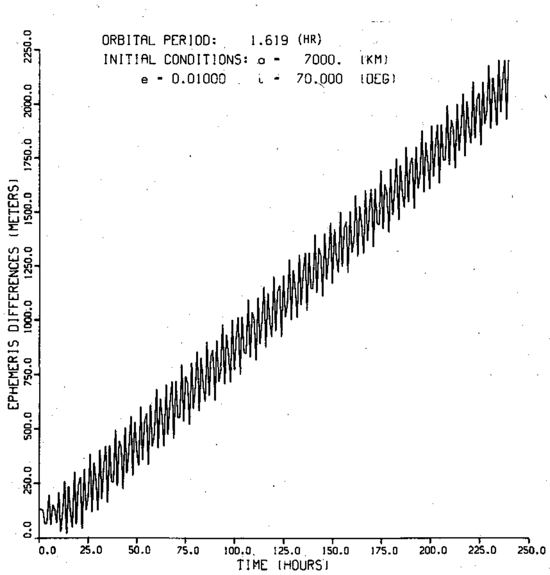

Fig. 2 GTDS (Cowell)/AOPP (order 1, J_2).

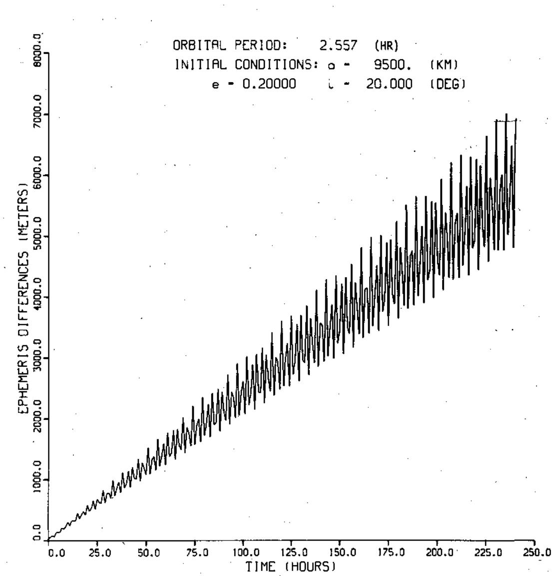

Fig. 4 GTDS (Cowell)/AOPP (order 1, J_2, J_3, J_4).

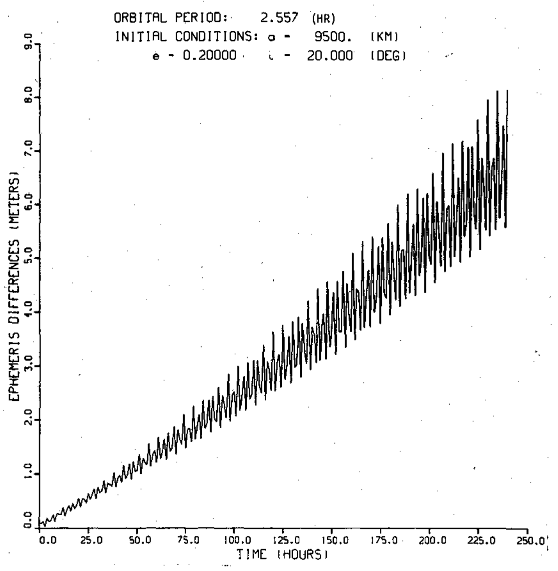

Fig. 5 GTDS (Cowell)/AOPP (order 2, J_2, J_3, J_4).

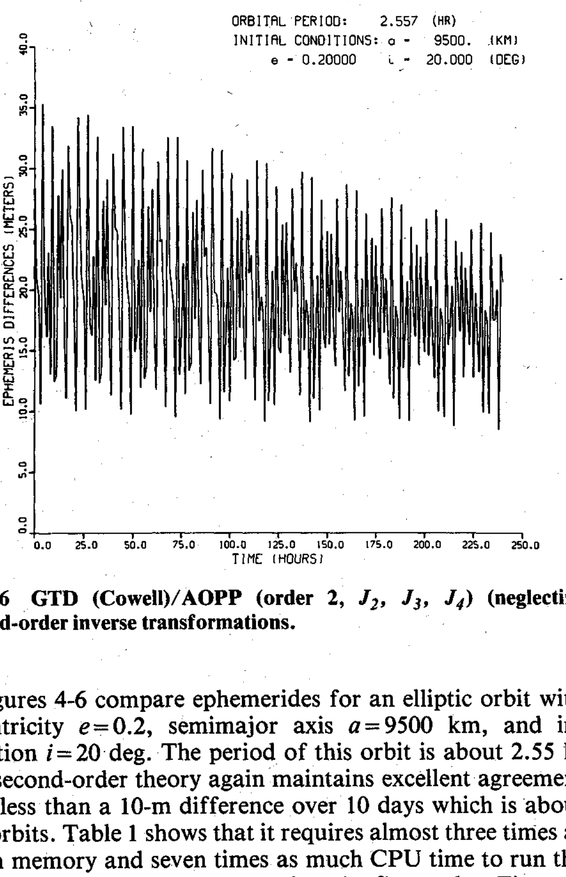

Fig. 6 GTD (Cowell)/AOPP (order 2, J_2, J_3, J_4) (neglecting second-order inverse transformations.

Figures 4-6 compare ephemerides for an elliptic orbit with eccentricity $e = 0.2$, semimajor axis $a = 9500$ km, and inclination $i = 20$ deg. The period of this orbit is about 2.55 h. The second-order theory again maintains excellent agreement with less than a 10-m difference over 10 days which is about 100 orbits. Table 1 shows that it requires almost three times as much memory and seven times as much CPU time to run the second-order theory as compared to the first order. Figure 6 shows that excellent results can be obtained by neglecting the second-order inverse transformations. For this comparison we use the full second-order transformations to establish highly accurate averaged elements and secular rates for these elements. But to the averaged orbit we apply only the first-order inverse transformations. Since the averaged elements are only computed once, while the inverse transformations must be computed at each time step, this brings the CPU time down to that of the first-order theory. Since for many real-time applications it is the averaged elements that are loaded into the memory of the microcomputer, this would produce an increase in accuracy with little increase in computer time or memory to produce the ephemerides.

V. Conclusions

The process of implementing an analytic satellite theory in operational ephemeris programs has been automated through the design of a program generator, called a meta-program. The meta-program provides users with the utility of a general-purpose program while at the same time tailoring the finished ephemeris programs efficiently, according to the specifications of the application. The meta-program forms a mechanized interface between the theory and the operational orbit programs, automating the last major link in the process of delivering a finished ephemeris program derived from a particular mathematical model. All of the options normally provided by a general-purpose program are provided by the meta-program with the efficiency of a single-purpose application program being preserved in the operational programs.

The concept of more than one ephemeris program emanating from a single automated source could provide benefits such as standardization of ephemeris programs and decreased software development costs. For instance, to facilitate the debugging process, it was decided to endow the meta-program with the ability of constructing an ephemeris program centered around the first-order torsion theory. This theory requires the first-order parallax transformations but completes the theory by executing the torsion transformation instead of the perigee and Delaunay transformations. This first-order theory was quite easy to implement with the machinery of the meta-program. A subroutine was coded merely for the torsion transformation and the meta-program was instructed to substitute this routine in place of the routines for executing the perigee and Delaunay transformations. The routines for input, output, series evaluation, coordinate transformations, solution to Kepler's equation, and all of the others needed for a finished program were already in place. Thus a new ephemeris program based on a different theory was quickly and automatically generated and assembled. Of course the meta-program could enable one to augment the mathematical model with other perturbations and with a minimum of effort produce an ephemeris program which reflects the new theory.

The options for designing the ephemeris programs are provided by the meta-program through a program template which is dictated by the user during a short session at a terminal. The program template could form the nucleus for a user-friendly environment for disseminating finished ephemeris programs. A user accessing the meta-program either by modem or through a computer network such as the Navy Laboratory Computer Network (NALCON) was envisioned. Within the environment provided on the host computer, a user would be instructed on how to complete the program template and run the meta-program. The user would ultimately pull back across the communication lines the freshly written ephemeris program for use on his own computing system.

One of the major objectives for the development of the ephemeris program was to demonstrate the feasibility of compressing the second-order theory into a program that could be run on microcomputers which typically have 64K bytes of memory. The intended purpose for the program was for applications in orbit research, satellite system design and analysis, and operational use in satellite systems. The first effort at implementing the second-order theory, which included the zonals J_2, J_3, J_4, required 147K bytes of memory to load the program and 1.2 s to complete the computations for one position. Improvements to the performance of the program were made through a number of revisions to the series storage and evaluation mechanism of the meta-program and by extensive simplifications to the series representing the theory. The procedure was to first construct the ephemeris programs as simply as possible to establish the correctness of the theory. Then step-by-step refinements were undertaken to improve the performance of the programs. For example storing the constant coefficient consisting of the numerator, denominator, J_3^n and J_4^n, for $n \geq 0$, as a single real constant reduced the memory requirements by using less memory for the coefficient and also by reducing the size of the series. While, for a particular program, this precludes one's ability to specify the zonal harmonics to include in the ephemeris program, this is not a limitation considering the meta-program's ability to construct separate programs with any combination of harmonics, within the framework of the existing theory. Judicious use of single-precision arrays in place of double-precision arrays also reduced the memory requirements without degrading the accuracy of the results. Run time performance was enhanced by evaluating the Delaunay transformation, except for the equation of the center ϕ, in the initialization part of the program outside the orbit propagation loop. With the use of the meta-program these modifications were passed through to the ephemeris programs easily. Ultimately the run time for the same program was reduced to 0.3 s per position and required only 41K bytes of memory to load it.

Appendix A

The partial derivatives of the secular Hamiltonian are:

$$\frac{\partial H}{\partial L} = \frac{\mu^2}{L^3} + \epsilon\frac{G}{p^2}\left(\frac{\alpha}{p}\right)^2 \eta^4\left(\frac{9}{4}s^2 - \frac{3}{2}\right) + \frac{\epsilon^2}{2}\left(\frac{\alpha}{p}\right)^4\frac{G}{p^2}\left(\left[\left[\left\{\frac{1575}{64}\frac{J_4}{J_2^2} + \frac{75}{64}\right\}\eta^6 + \frac{27}{4}\eta^5\right.\right.\right.$$

$$\left. - \left\{\frac{1575}{64}\frac{J_4}{J_2^2} - \frac{315}{64}\right\}\eta^4\right]s^4 + \left[\left\{-\frac{225}{8}\frac{J_4}{J_2^2} + \frac{15}{8}\right\}\eta^6 - 9\eta^5 + \left\{\frac{225}{8}\frac{J_4}{J_2^2} - \frac{45}{4}\right\}\eta^4\right]s^2 + \left\{\frac{45}{8}\frac{J_4}{J_2^2} - \frac{15}{8}\right\}\eta^6 + 3\eta^5 + \left\{-\frac{45}{8}\frac{J_4}{J_2^2} + \frac{45}{8}\right\}\eta^4\right)$$

$$\frac{\partial H}{\partial G} = \epsilon\frac{G}{p^2}\left(\frac{\alpha}{p}\right)^2 \eta^3\left(-3 + \frac{15}{4}s^2\right) + \frac{\epsilon^2}{2}\left(\frac{G}{p^2}\right)\left(\frac{\alpha}{p}\right)^4\left(\left[\left\{\frac{2835}{64}\frac{J_4}{J_2^2} + \frac{135}{64}\right\}\eta^5 + \frac{135}{8}\eta^4 + \left\{-\frac{5775}{64}\frac{J_4}{J_2^2} + \frac{1155}{64}\right\}\eta^3\right]s^4\right.$$

$$\left. + \left[\left\{-\frac{945}{16}\frac{J_4}{J_2^2} + \frac{27}{16}\right\}\eta^5 - \frac{99}{4}\eta^4 + \left\{\frac{1875}{16}\frac{J_4}{J_2^2} - \frac{645}{16}\right\}\eta^3\right]s^2 + \left\{\frac{135}{8}\frac{J_4}{J_2^2} - \frac{21}{8}\right\}\eta^5 + 9\eta^4 + \left\{-\frac{255}{8}\frac{J_4}{J_2^2} + \frac{165}{8}\right\}\eta^3\right)$$

$$\frac{\partial H}{\partial N} = \epsilon\frac{3}{2}\frac{G}{p^2}\left(\frac{\alpha}{p}\right)^2 c\eta^3 + \frac{\epsilon^2}{2}\frac{G}{p^2}\left(\frac{\alpha}{p}\right)^4 c\left(\left[\left\{\frac{315}{16}\frac{J_4}{J_2^2} + \frac{15}{16}\right\}\eta^5 + \frac{27}{4}\eta^4 + \left\{-\frac{525}{16}\frac{J_4}{J_2^2} + \frac{105}{16}\right\}\eta^3\right]s^2 + \left\{-\frac{45}{4}\frac{J_4}{J_2^2} + \frac{3}{4}\right\}\eta^5 - \frac{9}{2}\eta^4\right.$$

$$\left. + \left\{\frac{75}{4}\frac{J_4}{J_2^2} - \frac{15}{2}\right\}\eta^3\right)$$

$c = \cos I$

Appendix B

Due to the large number of transformations, only the transforming equations to first order will be provided. A prime represents a particular variable after the application of a transformation.

For the parallax transformation,

$$r = r' + \epsilon\left(\frac{\alpha}{p}\right)^2 p\left(\frac{1}{2} - \frac{3}{4}s^2 - \frac{1}{4}s^2\cos 2\theta\right)$$

$$R = R' + \epsilon\frac{1}{2}\frac{\Theta}{r^2}p\left(\frac{\alpha}{p}\right)^2 s^2\sin 2\theta$$

$$\theta = \theta' + \epsilon\left(\frac{\alpha}{p}\right)^2\left(-\frac{1}{4}c^2\bar{S}\cos 3\theta + \frac{1}{4}c^2\bar{C}\sin 3\theta + \left(\frac{3}{4} - \frac{7}{8}s^2\right)\sin 2\theta + \left(\frac{13}{4} - \frac{17}{4}s^2\right)\bar{S}\cos\theta - \frac{7}{4}c^2\bar{C}\sin\theta\right)$$

$$\Theta = \Theta' + \epsilon\left(\frac{\alpha}{p}\right)^2\Theta'\frac{s^2}{4}\left(-\bar{C}\cos 3\theta - \bar{S}\sin 3\theta - 3\cos 2\theta - 3\bar{C}\cos\theta + 3\bar{S}\sin\theta\right)$$

$$\nu = \nu' + \epsilon\left(\frac{\alpha}{p}\right)^2\frac{c}{4}\left(\bar{S}\cos 3\theta - \bar{C}\sin 3\theta - 3\sin 2\theta - 9\bar{S}\cos\theta + 3\bar{C}\sin\theta\right)$$

For the perigee transformation,

$$r = r' + \epsilon\frac{p}{1-5c^2}\left(\left(\frac{\alpha}{p}\right)^2\left\{\bar{C}\left[\left(\frac{35}{16}\frac{J_4}{J_2^2} + \frac{15}{16}\right)s^4 - \left(\frac{15}{8}\frac{J_4}{J_2^2} + \frac{7}{8}\right)s^2\right]\cos\theta + \bar{S}\left[-\left(\frac{35}{16}\frac{J_4}{J_2^2} + \frac{15}{16}\right)s^4 + \left(\frac{15}{8}\frac{J_4}{J_2^2} + \frac{7}{8}\right)s^2\right]\sin\theta\right\}\right.$$

$$\left. + \left(\frac{\alpha}{p}\right)\left(-\frac{5}{2}s^3 + 2s\right)\frac{J_3}{J_2^2}\sin\theta\right)$$

$$R=R'+\frac{\Theta p}{r^2}\frac{1}{1-5c^2}\Bigg(\Big(\frac{\alpha}{p}\Big)^2\Big\{\bar{S}\Big[\Big(-\frac{35}{16}\frac{J_4}{J_2^2}-\frac{15}{16}\Big)s^4+\Big(\frac{15}{8}\frac{J_4}{J_2^2}+\frac{7}{8}\Big)s^2\Big]\cos\theta+\bar{C}\Big[\Big(-\frac{35}{16}\frac{J_4}{J_2^2}-\frac{15}{16}\Big)s^4+\Big(\frac{15}{8}\frac{J_4}{J_2^2}+\frac{7}{8}\Big)s^2\Big]\sin\theta\Big\}$$

$$+\Big(\frac{\alpha}{p}\Big)\Big(2s-\frac{5}{2}s^3\Big)\frac{J_3}{J_2^2}\sin\theta\Bigg)$$

$$\theta=\theta'+\epsilon\frac{1}{1-5c^2}\Bigg(\frac{\alpha}{p}\Big\{\Big(s-\frac{5}{4}s^3\Big)(\bar{C}\cos2\theta+\bar{S}\sin2\theta)+(4s-5s^3)\cos\theta$$

$$+\frac{25}{4}s^3-\frac{17}{2}s+\frac{2}{s}\Big\}\frac{J_3}{J_2^2}+\Big(\frac{\alpha}{p}\Big)^2\Big\{(\bar{C}^2+\bar{S}^2)\Big(\Big[-\frac{35}{32}\frac{J_4}{J_2^2}-\frac{15}{32}\Big]s^4+\Big[\frac{15}{16}\frac{J_4}{J_2^2}+\frac{7}{16}\Big]s^2\Big)\sin2\theta$$

$$+(\bar{S}\cos\theta+\bar{C}\sin\theta)\Big(\Big[-\frac{35}{8}\frac{J_4}{J_2^2}-\frac{15}{8}\Big]s^4+\Big[\frac{15}{4}\frac{J_4}{J_2^2}+\frac{7}{4}\Big]s^2\Big)+\bar{C}\bar{S}\Big(\Big[\frac{35}{4}\frac{J_4}{J_2^2}+\frac{15}{4}\Big]s^4-\Big[\frac{25}{2}\frac{J_4}{J_2^2}+\frac{11}{2}\Big]s^2+\Big[\frac{15}{4}\frac{J_4}{J_2^2}+\frac{7}{4}\Big]\Big)\Big\}\Bigg)$$

$$+\epsilon\Big(\frac{1}{1-5c^2}\Big)^2\Bigg(\frac{\alpha}{p}\Big\{-25s^5+45s^3-20\Big\}\frac{J_3}{J_2^2}+\Big(\frac{\alpha}{p}\Big)^2\bar{C}\bar{S}\Big\{\Big(-\frac{175}{8}\frac{J_4}{J_2^2}-\frac{75}{8}\Big)s^6+\Big(\frac{325}{8}\frac{J_4}{J_2^2}+\frac{145}{8}\Big)s^4-\Big(\frac{75}{4}\frac{J_4}{J_2^2}+\frac{35}{4}\Big)s^2\Big\}\Bigg)$$

$$\Theta=\Theta'+\epsilon\Big(\frac{\theta'}{1-5c^2}\Big)\Bigg(\frac{\alpha}{p}\frac{J_3}{J_2^2}\bar{S}\Big(2s-\frac{5}{2}s^3\Big)+\Big(\frac{\alpha}{p}\Big)^2\Big[(\bar{C}^2-\bar{S}^2)\Big\{\Big(\frac{35}{16}\frac{J_4}{J_2^2}+\frac{15}{16}\Big)s^4-\Big(\frac{15}{8}\frac{J_4}{J_2^2}+\frac{7}{8}\Big)s^2\Big\}\Big]\Bigg)$$

$$\nu=\nu'+\frac{\epsilon c\bar{C}}{1-5c^2}\Bigg(\frac{\alpha}{p}\frac{J_3}{J_2^2}\Big(-\frac{2}{s}+\frac{15}{2}s\Big)+\Big(\frac{\alpha}{p}\Big)^2\Big\{\Big(\frac{35}{4}\frac{J_4}{J_2^2}+\frac{15}{4}\Big)s^2-\Big(\frac{15}{4}\frac{J_4}{J_2^2}+\frac{7}{4}\Big)\Big\}\Bigg)$$

$$+\frac{\epsilon c\bar{C}}{(1-5c^2)^2}\Bigg(\frac{\alpha}{p}\frac{J_3}{J_2^2}(20s-25s^3)+\Big(\frac{\alpha}{p}\Big)^2\bar{S}\Big\{\Big(-\frac{175}{8}\frac{J_4}{J_2^2}-\frac{75}{8}\Big)s^4+\Big(\frac{75}{4}\frac{J_4}{J_2^2}+\frac{35}{4}\Big)s^2\Big\}\Bigg)$$

For the Delaunay transformation,

$$\ell=\ell'+\epsilon\Big(\frac{\alpha}{p}\Big)^2\Big(\frac{\eta}{e}\Big\{1-\frac{3}{2}s^2\Big\}\sin f+\eta\Big\{\frac{1}{4}-\frac{3}{8}s^2\Big\}\sin2f\Big)$$

$$g=g'+\epsilon\Big(\frac{\alpha}{p}\Big)^2\Big(\phi\Big\{-3+\frac{15}{4}s^2\Big\}+\frac{1}{e}\Big\{-1+\frac{3}{2}s^2\Big\}\sin f+\Big\{-\frac{1}{4}+\frac{3}{8}s^2\Big\}\sin2f\Big)$$

$$h=h'+\epsilon\frac{3}{2}\Big(\frac{\alpha}{p}\Big)^2c\phi$$

$$L=L'+\epsilon\Big(\frac{\alpha}{p}\Big)^2\Theta\Big(\Big[-\frac{3}{4}\Big(\frac{1}{\eta}\Big)^3+\frac{1}{4}\frac{1}{\eta}+\frac{1}{2}+\Big\{\frac{9}{8}\Big(\frac{1}{\eta}\Big)^3-\frac{3}{8}\frac{1}{\eta}-\frac{3}{4}\Big\}s^2\Big]+\Big(\frac{1}{\eta}\Big)^3\Big[\frac{3}{2}s^2-1\Big]e\cos f+\Big(\frac{1}{\eta}\Big)^3\Big[\frac{3}{8}s^2-\frac{1}{4}\Big]e\cos2f\Big)$$

$\phi=f-\ell$ is the equation of the center.

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